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NOTE ON ISOGONAL TRANSFORMATION; PARTICULARLY ON OBTAINING CERTAIN SYSTEMS OF CURVES WHICH OCCUR IN THE STATICS OF POLYNOMIALS.*

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1. If $s, = \xi + i\eta$, be a function χ of $u, = X + iY$, then if

$$f(\xi, \eta) = 0 \tag{1}$$

be the equation of any path of s,

$$f(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0$$

is the equation of its image ($\mathfrak{r}\chi$ denoting the real part of χ and $i\mathfrak{p}\chi$ the imaginary part).

2. If u be a function of $z, = x + iy$,

$$f(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0$$

may be interpreted upon either the u-plane or the z-plane; that is, it may be regarded as an equation in X, Y or in x, y.

3. We may substitute for $\chi(u)$ its derivative with respect to z (or x), and the result will still be a function of z, and so, of course, of u.

The accent will be used to denote a differentiation with respect to x. When the quantity to which it is applied is a function of z, the accent may also be regarded as denoting a differentiation with respect to that letter.

4. If two curves

$$f_1(\xi, \eta) = 0, \quad f_2(\xi, \eta) = 0 \qquad (2)\ (3)$$

intersect at certain angles in the s-plane,

$$f_1(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0,$$
$$f_2(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0$$

* An elegant exposition of the subject here referred to may be found in an article by Félix Lucas entitled Statique des polynômes. Bulletin de la Société Mathématique de France, 1889.

See also the same author's memoirs in Comptes Rendus, especially 1868 (with a review upon them in 1870) and 1888.

intersect at like angles in the u-plane (or z-plane), provided ds/du (or ds/dz) neither vanish nor become infinite at the points of intersection of (2) and (3).

5. $s = u$.

The images of the orthogonal systems

$$\xi^2 + \eta^2 = (\text{constant})^2, \tag{4}$$

$$\eta/\xi = \text{constant} \tag{5}$$

are the orthogonal systems

$$X^2 + Y^2 = (\text{constant})^2, \tag{6}$$

$$Y/X = \text{constant}. \tag{7}$$

In the z-plane,* supposing u to be a polynomial in z of order m, equation (6) denotes a system of curves of order $2m$ which have been called *lemniscates of the mth order*, or *Cassinoids*, since the continued product of the moduli of the factors of u is constant for any given curve of the system. Equation (7) denotes a system of curves of order m, each having m hyperbolic branches setting out from the zeros of u, and have been called *hyperbolas of the mth order*. All asymptotes pass through the centre of mean position of the zeros, thus dividing the z-plane into $2m$ equal angular compartments ; for this reason the curves have been also called *stelloids*.

6. Regarding the zeros of the polynomial u as fixed points of unit mass which act upon the variable point inversely as their distances from it, equation (6) represents a system of *equipotential lines* in the z-plane. Upon the same suppositions, equation (7) represents a system of *lines of force*.

7. $s = \dfrac{\partial^n u}{\partial x^n}$.

The image of any curve

$$f(\xi, \eta) = 0$$

is

$$f\left[\frac{\partial^n X}{\partial x^n}, \frac{\partial^n Y}{\partial x^n}\right] = 0. \tag{8}$$

In particular, this shows that because the systems (4) and (5) are orthogonal, so are the systems

$$\left[\frac{\partial^n X}{\partial x^n}\right]^2 + \left[\frac{\partial^n Y}{\partial x^n}\right]^2 = (\text{constant})^2, \tag{9}$$

$$\frac{\partial^n Y}{\partial x^n} \Big/ \frac{\partial^n X}{\partial x^n} = \text{constant}. \tag{10}$$

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* Journal de l'École Polytechnique, 1879: Géométrie des polynômes.
Holzmüller, Theorie der isogonalen Verwandschaften, S. 202, 204 *), 205 **).
Annals of Mathematics, Vol. iv, p. 73.

If $n = 1$, the curves (9) become

$$X'^2 + Y'^2 = (\text{constant})^2, \tag{11}$$

which are *lines of equal expansion* for the function u; the orthogonal trajectories are

$$Y'/X' = \text{constant}. \tag{12}$$

8. $s = \dfrac{\partial^n \log u}{\partial x^n}$.

The image of any curve

$$f(\xi, \eta) = 0$$

is

$$f\left[\frac{\partial^n \log R}{\partial x^n}, \frac{\partial^n \theta}{\partial x^n}\right] = 0. \tag{13}$$

When $n = 1$, s becomes u'/u, and (13)

$$f(\log' R, \theta') = 0.$$

Equation (4) now transforms into

$$\frac{X'^2 + Y'^2}{X^2 + Y^2} = (\text{constant})^2. \tag{14}$$

These curves are *isodynamic lines*, upon the hypotheses made in § 6. $\xi, -\eta$ are equal to the total component forces along the x- and the y-axis respectively.

From equation (5) it follows that the system

$$\frac{XY' - X'Y}{XX' + YY'} = \text{constant} \tag{15}$$

is orthogonal to (14). These curves are *lines of parallel action.*

Lucas's generalization of Rolle's theorem. Since

$$\xi + i\eta = \frac{u'}{u}$$

it is clear that a variable point in the z-plane will be in equilibrium when, and only when, it coincides with a zero of the derived equation $u' = 0$.

Every closed convex contour surrounding the roots of an algebraic equation surrounds also the roots of the derived equation.

9. *Application to Hydrodynamics.* A steady irrotational motion is supposed to take place in planes parallel to xy.

Let $X =$ constant be the lines of equal velocity-potential; then $Y =$ constant are the stream-lines, (11) the lines of equal velocity, (12) the lines along which the direction of flow is constant.

Here and in § 7, the only condition imposed upon u is that it be a function of z; it may, therefore, be replaced by any given function of u, and the four systems of curves just mentioned will be altered accordingly. If $\log u$ be substituted for u, these systems assume the forms (6), (7), (14), and (15), respectively. Suppose, in a fluid of unlimited extent, line sources, each of the same strength and perpendicular to the plane xy, to pass through the zeros of a polynomial u. For this motion (6) are the lines of equal velocity-potential, (7) the stream-lines, (14) the lines of equal velocity, and (15) the lines along which the direction of flow is constant.